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1994 J. Phys. A: Math. Gen. 27 4933

(http://iopscience.iop.org/0305-4470/27/14/021)

J. Phys. A: Math. Gen. 27 (1994) 4933–4943. Printed in the UK

Time-dependent propagator with point interaction

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Received 21 June 1993

Abstract. We compute the time-dependent Schrödinger propagator with a point interaction in dimension $n \leqslant 3$ including the new cases of $n = 2$ and the most general interaction supported by a point for $n = 1$. We also give the small-time asymptotics for $n \leqslant 3$. The case $n = 2$ has the peculiarity of involving logarithmic terms in the expansion.

Recently the time-dependent propagator was computed explicitly for δ-potentials in dimension $n = 1$ (Segal 1972, Schulman 1986, Manoukian 1989) (for the corresponding heat kernel see Gaveau and Schulman (1986)) and $n = 3$ (Scarletti and Teta 1990). In this paper we report a similar computation for the most general local (singular) perturbation concentrated at one point ($x = 0$) in dimension $n \leqslant 3$. We also present a small-time asymptotic expression for the propagator.

As a new result we treat the four-parameter family (which corresponds to a mixture of δ- and δ'-potentials) in dimension $n = 1$, and the two-dimensional case $n = 2$, where we have a double integral representation which is suitable, e.g., to exhibit explicitly the small-time asymptotic expansion. For completeness, we shall derive in a simpler way the known integral expressions for the time-dependent propagator in the case of the δ-potential for $n = 1$ and $n = 3$.

For $n = 2, 3$ one has a one-parameter family which can be parametrized by $\alpha \in [-\infty, \infty]$, with α being related to the inverse scattering length (Albeverio *et al* 1988). For $n = 1$ the situation is more rich and there is a four-parameter family (Chernoff and Hughes 1993, cf also Segal 1972), characterized by the boundary conditions at $x = 0$:

$$\varphi(0_+) = \omega a \varphi(0_-) + \omega b \varphi'(0_-) \tag{1}$$

$$\varphi'(0_+) = \omega c \varphi(0_-) + \omega d \varphi'(0_-)$$

with $\omega \in \mathbb{C}$; $a, b, c, d \in \mathbb{R}$, satisfying $|\omega| = 1$ and $ad - bc = 1$ (four independent real coordinates).

Two particular subfamilies: $b = 0$, $\omega = a = d = 1$, $c = \alpha$, and $c = 0$, $\omega = a = d = 1$, $b = \alpha$, respectively, corresponding to extensions denoted usually as $-\Delta + \alpha\delta$ or $-\Delta + \alpha\delta'$, have been extensively studied, see Albeverio *et al* (1988). In these two cases $\alpha = \pm\infty$

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0305-4470/94/144933+11$19.50

corresponds to the Dirichlet boundary conditions at $x = 0$ in the case of a δ-potential and to the Neumann conditions in the case of a δ'-potential.

An equivalent characterization of the perturbed operators is provided by the Green function

$$\hat{G}_\lambda(x, y) = G_\lambda(x - y) + \gamma(\lambda; x, y) G_\lambda(x) G_\lambda(y) \tag{2}$$

where

$$\gamma^\alpha(\lambda) = \begin{cases} \dfrac{2\sqrt{\lambda}}{D(\lambda)}[-c - B(x, y)\sqrt{\lambda}/2 + b\lambda\,\mathrm{sgn}(xy)] & \text{for } n = 1 \\ 4\pi(\alpha + \ln\lambda)^{-1} & \text{for } n = 2 \\ 4\pi(\sqrt{\lambda} + 4\pi\alpha)^{-1} & \text{for } n = 3 \end{cases} \tag{3}$$

where

$$D(\lambda) = [b\lambda + (a + d)\sqrt{\lambda} + c] \tag{4}$$

and

$$2B(x, y) = a+d-\omega-\bar{\omega}+(d-a-\omega+\bar{\omega})\,\mathrm{sgn}\, x+(d-a+\omega-\bar{\omega})\,\mathrm{sgn}\, y-(a+d-\omega-\bar{\omega})\,\mathrm{sgn}(xy). \tag{5}$$

Above, $G_\lambda(x, y) = G_\lambda(x - y)$ is the free Green function with

$$G_\lambda(x) = \begin{cases} \dfrac{1}{2\sqrt{\lambda}} \mathrm{e}^{-\sqrt{\lambda}|x|} & \text{for } n = 1 \\ \dfrac{1}{2\pi} K_0(\sqrt{\lambda}|x|) & \text{for } n = 2 \\ \dfrac{1}{4\pi|x|} \mathrm{e}^{-\sqrt{\lambda}|x|} & \text{for } n = 3. \end{cases} \tag{6}$$

and K_0 denoting the Macdonald function (modified Bessel function of the third kind) of order 0.

One easily recovers the known expressions in the case of $\alpha\delta$ and $\alpha\delta'$, as then $B(x, y) = 0$.

The perturbed spectrum (cf Albeverio *et al* 1988, Chernoff and Hughes 1993) consists of continuous spectrum $[0, \infty)$ and the discrete spectrum with one or two simple eigenvalues under the folllowing conditions on the parameters:

$$E = \begin{cases} -\left(\dfrac{c}{a+d}\right)^2 & \text{for } b = 0 & \dfrac{c}{a+d} < 0 & n = 1 \\ -A_-^2 & \text{for } b \neq 0 & A_- < 0 \leqslant A_+ & n = 1 \\ -A_-^2, -A_+^2 & \text{for } b \neq 0 & A_- < A_+ < 0 & n = 1 \\ -4\mathrm{e}^{-2\alpha} & \text{for } \alpha \in \mathbb{R} & n = 2 & \\ -(4\pi\alpha)^2 & \text{for } \alpha < 0 & n = 3 & \end{cases} \tag{7}$$

where

$$A_{\pm} = (a+d)/2b \pm \sqrt{(a-d)^2+4}\,/2|b|. \tag{8}$$

(Note that we always have $A_- < A_+$.) The corresponding (unnormalized) eigenvectors are given respectively by

$$\Psi(x) = \begin{cases} [\theta(-x)+a\theta(x)]G_{(c/a+d)^2}(x) & \text{for } b=0,\ \frac{c}{a+d}<0,\ n=1 \\ [\theta(-x)+\omega\theta(x)(a+bA_-)]G_{A_-^2}(x) & \text{for } b\neq 0,\ A_-<0\leqslant A_+,\ n=1 \\ [\theta(-x)+\omega\theta(x)(a+bA_{\pm})]G_{A_{\pm}^2}(x) & \text{for } b\neq 0,\ A_-<A_+<0,\ n=1 \\ G_{4\mathrm{e}^{-2\alpha}}(x) & \text{for } \alpha\in\mathbb{R},\ n=2 \\ G_{(4\pi\alpha)^2}(x) & \text{for } \alpha<0,\ n=3 \end{cases} \tag{9}$$

where $\theta(x) = \frac{1}{2}(1+\operatorname{sgn} x)$ is the step function. In the following we shall need the square norms

$$\|\Psi\|^2 = \begin{cases} -\frac{1}{2}(a+d)^3(a^2+1)c^{-3} & \text{for } b=0 \qquad n=1 \\ -(1+(a+bA_{\pm})^2)(2A_{\pm})^{-3} & \text{for } b\neq 0 \qquad n=1 \\ -\pi^{-2}\alpha^{-1}/16 & \text{for } \alpha<0 \qquad n=3\,. \end{cases} \tag{10}$$

The time-dependent propagator $\hat{P}_t(x,y)$ for the perturbed Schrödinger operator can be obtained from the Green function $\hat{G}_t(x,y)$. First, one calculates $\hat{P}(z;x,y)$ for $z=t>0$ as the inverse Laplace transform of $\hat{G}_\lambda(x,y)$, obtaining the integral kernel of the 'heat' semigroup. Next, one analytically continues $\hat{P}(z;x,y)$ to $z=\mathrm{i}t$, $t>0$, which should provide $\hat{P}_t(x,y)$, the integral kernel of the corresponding unitary evolution operator. This second step may require special attention.

The integrals needed for the computation of $\hat{P}(z;x,y)$ as the inverse Laplace transform of $\hat{G}_\lambda(x,y)$ in the case of the δ-potential for $n=1$ and for $n=3$, are tabulated (e.g. Erdelyi 1954). But, in fact, one obtains the answer in the above cases, and also in the (new) case of general self-adjoint extensions for $n=1$, by using a simple property of the free Green's function. Namely, the second term in (2) can be rewritten (if $\operatorname{Re}\lambda$ is sufficiently large) for $n=1$, $b=0$, as

$$\begin{aligned} &\left\{-\frac{B}{a+d}+\frac{c/(a+d)(B(x,y)/(a+d)-1)}{\sqrt{\lambda}+[c/(a+d)]}\right\}G_\lambda(|x|+|y|) \\ &\qquad = -\frac{B}{a+d}G_\lambda(|x|+|y|)+\frac{c}{a+d}\left(\frac{B(x,y)}{a+d}-1\right) \\ &\qquad\quad \times\int_0^\infty \mathrm{e}^{-[c/(a+d)]u}G_\lambda(u+|x|+|y|)\,\mathrm{d}u \end{aligned}$$

for $n=1$ and $b\neq 0$ as

$$\begin{aligned} &\left(\operatorname{sgn}(xy)+\frac{M_+}{\sqrt{\lambda}+A_+}-\frac{M_-}{\sqrt{\lambda}+A_-}\right)G_\lambda(|x|+|y|) \\ &\qquad = \operatorname{sgn}(xy)G_\lambda(|x|+|y|)+\int_0^\infty (M_+\mathrm{e}^{-A_+u}-M_-\mathrm{e}^{-A_-u})G_\lambda(u+|x|+|y|)\,\mathrm{d}u \end{aligned}$$

where, with $A_\pm$ defined in (8),

$$M_\pm = \frac{1}{b(A_+ - A_-)}\{A_\pm[B(x,y)/2 + (a+d)\,\mathrm{sgn}(xy)] - (\mathrm{sgn}(xy)+1)c\} \tag{11}$$

and, for $n = 3$, as

$$\frac{1}{2\pi(4\pi\alpha + \sqrt{\lambda})}\frac{\mathrm{e}^{-\sqrt{\lambda}(|x|+|y|)}}{2\sqrt{\lambda}} = \frac{1}{2\pi|x||y|}\frac{\mathrm{e}^{-\sqrt{\lambda}(|x|+|y|)}}{2\sqrt{\lambda}} - \frac{2\alpha}{|x||y|}\int_0^\infty \mathrm{e}^{-4\pi\alpha u}\frac{\mathrm{e}^{-\sqrt{\lambda}(u+|x|+|y|)}}{2\sqrt{\lambda}}\,\mathrm{d}u.$$

Next, by using the fact that the inverse Laplace transform of the free Green function $G_\lambda(x)$ is known to be the 'heat' kernel

$$P(t;x) = (4\pi t)^{-n/2}\mathrm{e}^{-|x|^2/4t} \tag{12}$$

by linearity of the inverse Laplace transform, we immediately obtain the following formulae

$$\hat{P}(t;x,y) = P(t,x-y) + \begin{cases} -\dfrac{B(x,y)}{a+d}P(t;|x|+|y|) + \dfrac{c}{a+d}\left(\dfrac{B(x,y)}{a+d}-1\right) & \\ \quad\times\displaystyle\int_0^\infty \mathrm{e}^{-[c/(a+d)]u}P(t;u+|x|+|y|) & \text{for } n=1,\ b=0 \\ \mathrm{sgn}(xy)P(t,|x|+|y|) + \displaystyle\int_0^\infty (M_+\mathrm{e}^{-A_+u} - M_-\mathrm{e}^{-A_-u}) & \\ \quad\times P(t,u+|x|+|y|)\,\mathrm{d}u & \text{for } n=1,\ b\neq 0 \\ \dfrac{2t}{|x||y|}P(t;|x|+|y|) - \dfrac{8\pi\alpha t}{|x||y|}\displaystyle\int_0^\infty \mathrm{e}^{-4\pi\alpha u} & \\ \quad\times P(t;u+|x|+|y|)\,\mathrm{d}u & \text{for } n=3 \end{cases} \tag{13}$$

where $B(x,y)$ was defined in (5).

Now we discuss the analytic continuation of $\hat{P}(t;x,y)$, $t>0$, to imaginary time $z = \mathrm{i}t$, $t>0$. This is not immediate since the integrals depend on parameters (such as α for $n=3$) and the very existence of the integrals and, after that, analytic continuation has to be established, e.g. when $\alpha < 0$. (In Gaveau and Schulman (1986), the case $n=1$ with the δ-potential analytic continuation was discussed via analytic continuation in the mass parameter.) The continuation is certainly possible when $c/(a+d)$, A_+, A_- and α are positive, as then the integrals converge absolutely (for large u the exponential term dominates the oscillatory term in $P(\mathrm{i}t;u+|x|+|y|)$). Moreover, if $n=1$ and $b=c=0$, or $n=3$ and $\alpha=0$, the terms containing integrals disappear and the surviving terms have obvious analytic continuation. Also, if $n=1$ and $b\neq 0$, and either A_+ or A_- vanish (and thus also M_+ or M_- vanish), the corresponding integrals disappear. Finally, any time one of the constants $c/(a+d)$, A_+, A_- or α becomes negative, extending the procedure developed by Scarlatti and Teta (1990) (based on the identity $\int_{-\infty}^{\infty}\mathrm{e}^{-z^2}\,\mathrm{d}z = \sqrt{\pi}$), we can extract explicitly the contribution due to a bound state, and change at the same time $u \to -u$ in the exponent and the terms $u+|x|+|y|$ into $u-|x|-|y|$. Altogether, we obtain the following results.

For $n = 1$ and $b = 0$, the propagator of the Schrödinger equation with a point interaction is given by

$$\hat{P}(\mathrm{i}t; x, y) = P_t(x-y) - \frac{B(x,y)}{a+d} P_t(|x|+|y|) + \begin{cases} \dfrac{c}{a+d}\left(\dfrac{B(x,y)}{a+d} - 1\right)\displaystyle\int_0^\infty \exp\left(-\frac{c}{a+d}u\right) \\ \quad \times P_t(u+|x|+|y|)\,\mathrm{d}u & \text{for } \dfrac{c}{a+d} > 0 \\ 0 & \text{for } c = 0 \\ \exp(Et)\dfrac{\Psi(x)\Psi(y)}{\|\Psi\|^2} - \dfrac{c}{a+d}\left(\dfrac{B(x,y)}{a+d} - 1\right) \\ \quad \times \displaystyle\int_0^\infty \exp\left(\frac{c}{a+d}u\right) P_t(u-|x|-|y|)\,\mathrm{d}u & \text{for } \dfrac{c}{a+d} < 0 \end{cases} \tag{14}$$

where $E = (c/(a+d))^2$, $\Psi(x) = [\theta(-x) + a\theta(x)]G_{(c/(a+d))^2}(x)$, $\|\Psi\|^2 = -(a(a+d)^3/(8c^2))$, $P(t;x)$ is given by (12) and B by (5).

For $n = 1$ and $b \neq 0$ we have, depending on the position of $A_- < A_+$ with respect to zero,

$$\hat{P}_t(x,y) = P_t(x-y) + \mathrm{sgn}(xy)P_t(|x|+|y|) + \begin{cases} \displaystyle\int_0^\infty (M_+\mathrm{e}^{-A_+u} - M_-\mathrm{e}^{-A_-u}) \\ \quad \times P_t(u+|x|+|y|)\,\mathrm{d}u & \text{for } 0 < A_- < A_+ \\ \displaystyle\int_0^\infty M_+\mathrm{e}^{-A_+u} P_t(u+|x|+|y|)\,\mathrm{d}u & \text{for } A_- = 0 \\ \dfrac{\mathrm{e}^{E_-t}\Psi_-(x)\Psi_-(y)}{\|\Psi_-\|^2} + \displaystyle\int_0^\infty M_+\mathrm{e}^{-A_+u} P_t(u+|x|+|y|)\,\mathrm{d}u \\ \quad + \displaystyle\int_0^\infty M_-\mathrm{e}^{A_-u} P_t(u-|x|-|y|)\,\mathrm{d}u & \text{for } A_- < 0 < A_+ \\ \dfrac{\mathrm{e}^{E_-t}\Psi_-(x)\Psi_-(y)}{\|\Psi_-\|^2} \\ \quad + \displaystyle\int_0^\infty M_-\mathrm{e}^{A_-u} P_t(u-|x|-|y|)\,\mathrm{d}u & \text{for } A_+ = 0 \\ \dfrac{\mathrm{e}^{E_-t}\Psi_-(x)\Psi_-(y)}{\|\Psi_-\|^2} + \dfrac{\mathrm{e}^{E_+t}\Psi_+(x)\Psi_+(y)}{\|\Psi_+\|^2} - \displaystyle\int_0^\infty (M_+e^{A_+u} - M_-e^{A_-u}) \\ \quad \times P_t(u-|x|-|y|)\,\mathrm{d}u & \text{for } A_- < A_+ < 0 \end{cases} \tag{15}$$

where $E_\pm = A_\pm^2$, $\Psi_\pm(x) = [\theta(-x) + \omega(a + bA_\pm)\theta(x)]G_{A_\pm^2}(x)$, $\|\Psi_\pm\|^2 = -[1 + (a + bA_\pm)^2]/(2A_\pm)$ and $A_\pm$ and $M_\pm$ are defined respectively by (8) and (11).

Finally, for $n = 3$, we have

$$\hat{P}_t(x,y) = P_t(x-y) + \frac{2it}{|x||y|} P_t(|x|+|y|) + \begin{cases} -\frac{8\pi\alpha i t}{|x||y|} \int_0^\infty e^{-4\pi\alpha u} P_t(u+|x|+|y|)\, du & \text{for } \alpha > 0 \\ 0 & \text{for } \alpha = 0 \\ e^{iEt} \frac{\Psi(x)\Psi(y)}{\|\Psi\|^2} + \frac{8\pi\alpha i t}{|x||y|} \int_0^\infty e^{4\pi\alpha u} P_t(u-|x|-|y|)\, du & \text{for } \alpha < 0 \end{cases} \tag{16}$$

where now $E = 4\pi\alpha^2$, $\Psi(x) = G_{(4\pi\alpha)^2}(x)$ and $\|\Psi\|^2 = \pi^{-2}\alpha^{-1}/16$.

We note that formula (16), after integration by parts, coincides with (5) in Scarlatti and Teta (1990).

Equations (14)–(16) complete the computation of $\hat{P}_t(x,y)$ for all the cases in $n = 1$ and $n = 3$. The two-dimensional case ($n = 2$) turns out to be more complicated and we do not have a method similar to that used for $n = 1$ or $n = 3$. In particular, the functions $K_0(\sqrt{\lambda}|x|)$ do not possess a simple multiplicative property, which would simplify the second term in (2) and lead to expressions such as (13) for $\hat{P}(t;x,y)$.

Calculating directly, we write the inverse Laplace transform of the second term in (2) as a convolution and, using formula (64) on p 285 of Erdelyi (1954), we obtain the following double integral representation:

$$\hat{P}(t;x,y) = P(t;x-y) + \frac{1}{2\pi}\int_0^t d\tau \int_0^\infty du\, \frac{\exp(-\alpha u)(t-\tau)^{u-1}}{\Gamma(u)\tau} \times \exp\left(-\frac{|x|^2+|y|^2}{4\tau}\right) K_0\left(\frac{|x||y|}{2\tau}\right). \tag{17}$$

It is convenient to give two equivalent expressions. First, after the replacement of τ by t/z, we can write (17) as

$$\hat{P}(t;x,y) = P(t;x-y) + \frac{1}{2\pi}\int_0^\infty du\, \frac{t^{u-1}e^{-\alpha u}}{\Gamma(u)} \int_1^\infty dz\, (z-1)^{u-1} z^{-u} \times \exp\left(-\frac{|x|^2+|y|^2}{4t} z\right) K_0\left(\frac{|x||y|}{2t} z\right). \tag{18}$$

To the best of our knowledge, the integral over z is not known explicitly except for the case $|x|^2 + |y|^2 = 2|x||y|$, i.e. the case $|x| = |y|$ when the points x and y are equally distant from the origin, at which the δ-potential is placed . Then (18) becomes, cf Gradshteyn and Ryzhik (1965), p 715, equation (7)

$$\int_0^\infty du\, t^{-1} e^{-\alpha u} \pi^{-1/2} |x|^u |y|^u G_{23}^{30}\left(\frac{|x||y|}{t}\middle| \begin{matrix} 0 & \frac{1}{2} & -u \\ -u & -u & -u \end{matrix}\right)$$

where G_{23}^{30} is the Meijer's G-function (Gradshteyn and Ryzhik 1965, p 1068). Second, by the additional change of variables $z \to r+1$ in (18) we obtain

$$\hat{P}(t;x,y) = P(t;x-y) + \frac{1}{2\pi}\int_0^\infty du\, \frac{t^{u-1}e^{-\alpha u}}{\Gamma(u)} \int_0^\infty dr\, r^{u-1}(r+1)^{-u} \times \exp\left(-\frac{|x|^2+|y|^2}{4t}(r+1)\right) K_0\left(\frac{|x||y|}{2t}(r+1)\right). \tag{19}$$

Now, by extracting the large-z behaviour

$$K_0(z) \sim \mathrm{e}^{-z}\sqrt{\pi/2z}$$

we introduce

$$\widetilde{K}_0(z) = \mathrm{e}^{z}\sqrt{2z/\pi}\,K_0(z) \tag{20}$$

which satisfies $\widetilde{K}_0(z) \sim 1$ as $z \to \infty$. Thus, (19) can be rewritten as

$$\hat{P}(t;x,y) = P(t;x-y) + (4\pi t|x|\,|y|)^{-1/2}\exp\left(-\frac{(|x|+|y|)^2}{4t}\right)\int_0^\infty \mathrm{d}u\,\frac{t^u\mathrm{e}^{-\alpha u}}{\Gamma(u)}\cdot \\ \times \int_0^\infty \mathrm{d}r\,r^{u-1}(r+1)^{-u-1/2}\exp\left(-\frac{(|x|+|y|)^2}{4t}r\right)\widetilde{K}_0\left(\frac{|x||y|}{2t}(r+1)\right). \tag{21}$$

This expression will be used later on to obtain the small-time asymptotic expansion for $\hat{P}(t;x,y)$. Moreover, it can be analytically continued to imaginary time. Namely, the term containing integrals makes sense for $\mathrm{Re}\,t \geqslant 0$. Indeed, $|(r+1)^{-u^{-1/2}}r^{u-1}| < r^{-3/2}$ for $r \geqslant 1$. In addition, it is analytic for $\mathrm{Re}\,t > 0$ and continuous for $\mathrm{Re}\,t \geqslant 0$. Therefore we have the following result.

For $n = 2$ the time-dependent propagator of the Schrödinger equation with a point interaction is given by

$$\hat{P}_t(x,y) = P_t(x-y) + (4\pi\mathrm{i}t|x||y|)^{-1/2}\exp\left(\mathrm{i}\frac{(|x|+|y|)^2}{4t}\right)\int_0^\infty \mathrm{d}u\,\frac{\mathrm{i}t^u\mathrm{e}^{-\alpha u}}{\Gamma(u)} \\ \times \int_0^\infty \mathrm{d}r\,r^{u-1}(r+1)^{-u-1/2}\exp\left(\mathrm{i}\frac{(|x|+|y|)^2}{4t}r\right)\widetilde{K}_0\left(\frac{|x||y|}{2\mathrm{i}t}(r+1)\right). \tag{22}$$

Now we shall give the small-time asymptotic expansion as $t \to 0$ of the time-dependent Schrödinger propagator with a point interaction, referring to Albeverio *et al* (1994) for the details. We start with dimension $n = 3$. Define the coefficients $a_{k+1} = g_k(0)/\phi'(0)$, $k = 1, 2, \ldots; a \in \mathbb{R}$, with

$$g_0(u) = g(u) \qquad g_k(u) = \frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{g_{k-1}(u)}{\phi'(u)}\right) \tag{23}$$

where

$$g(u) = \mathrm{e}^{-4\pi\alpha u} \qquad \phi(u) = \tfrac{1}{4}(u+R)^2 \qquad \phi'(u) = \tfrac{1}{2}(u+R).$$

In particular we have

$$a_1 = \frac{g(0)}{\phi'(0)} = \frac{2}{R} \qquad a_2 = \frac{g_1(0)}{\phi'(0)} = \frac{-4(4\pi\alpha R+1)}{R^3}. \tag{24}$$

Then $\hat{P}_t(x, y)$ for $n = 3$, defined by (16), has the asymptotic expansion

$$\hat{P}_t(x, y) \sim (4\pi \mathrm{i}t)^{-3/2} \exp\left(\mathrm{i}\frac{|x-y|^2}{4t}\right) \frac{2\pi}{|x||y|}(4\pi \mathrm{i}t)^{-1/2} \exp\left(\mathrm{i}\frac{|x|^2+|y|^2}{4t}\right) + 2\alpha(4\pi \mathrm{i}t)^{-1/2} \sum_{k=1}^{\infty} \exp\left(\mathrm{i}\frac{|x|^2+|y|^2}{4t}\right) a_k(\mathrm{i}t)^k. \tag{25}$$

To compare this with a similar expansion of the heat kernel in dimension $n = 3$

$$\hat{P}(t; x, y) \sim P(t; x-y)\left\{1 + \sum_{k=1}^{\infty} a_k t^k\right\}$$

we can consider the case $\alpha = 0$. We have

$$\hat{P}(t; x, y) = P(t; x-y)\left(1 + \frac{2t}{|x||y|} \exp\left(-\frac{|x||y| + \langle x, y\rangle}{2t}\right)\right). \tag{26}$$

Thus, if x is opposite to y with respect to the origin (in the sense that $y = -sx$ for some $s > 0$), we have $|x||y| + \langle x, y\rangle = 0$ and

$$\hat{P}(t; x, y) = P(t; x-y)\left(1 + \frac{2t}{|x||y|}\right) \tag{27}$$

while if it is not, we have $|x||y| + \langle x, y\rangle > 0$ and

$$\hat{P}(t; x, y) \sim P(t; x-y). \tag{28}$$

Therefore, we have a free propagation which is 'perturbed' only when x is opposite to y with respect to the origin. In the Schrödinger case instead, the perturbation is always felt and, for $\alpha = 0$, (16) consists of two terms. The (main) first term corresponds in diagrammatic notation to a simple straight path joining x and y, while the second term corresponds to the broken path joining x with 0 and then with y. Similar considerations can be extended to higher terms $k \geqslant 2$ in (25). Thus, even though the standard perturbation theory does not work for our operators ($-\Delta$ perturbed by a singular potential 'V' supported at 0), it still makes sense to regard the exact propagator as the free one plus 'perturbative corrections'.

We note that the two paths above can be viewed as classical solutions to the Newton equation

$$\frac{\mathrm{d}^2\gamma(s)}{\mathrm{d}s^2} = -\nabla\text{'}V\text{'}(\gamma(s)) \qquad 0 \leqslant s \leqslant t \qquad \gamma(0) = x \qquad \gamma(t) = y$$

(with the potential vanishing everywhere but the origin).

Next we give the asymptotic behaviour $\hat{P}_t(x, y)$ in dimension $n = 1$ with boundary conditions (1):

$$\hat{P}_t(x, y) \sim (4\pi \mathrm{i}t)^{-1/2}\left\{\exp\left(\frac{\mathrm{i}\,|x-y|^2}{4t}\right) - \frac{B(x; y)}{a+d} \exp\left(\frac{i(|x|+|y|)^2}{4t}\right)\right\} + (4\pi)^{-1/2} \frac{c}{a+d}\left(\frac{B(x; y)}{a+d} - 1\right) \exp\left(\frac{\mathrm{i}(|x|+|y|)^2}{4t}\right) \sum_{k=1}^{\infty} b_k(\mathrm{i}t)^{k-(1/2)} \tag{29}$$

if $b = 0$ and

$$\hat{P}_t(x, y) \sim (4\pi \mathrm{i}t)^{-1/2} \left\{ \exp\left(\frac{|x-y|^2}{4t} \right) + \mathrm{sgn}(xy) \exp\left(\frac{\mathrm{i}(|x|+|y|)^2}{4t} \right) \right\}$$
$$+ (4\pi)^{-1/2} \sum_{k=1}^{\infty} [M_+ b_k(A_+) - M_- \tilde{b}_k(A_-)](\mathrm{i}t)^{k-(1/2)} \tag{30}$$

if $b \neq 0$. Here, we use the following notation:

$$b_k = 2\frac{g_{k-1}(0)}{|x|+|y|} \qquad \tilde{b}_k = \begin{cases} b_k & c/(a+d) \neq 0 \\ 0 & c/(a+d) = 0 \end{cases} \tag{31}$$
$$g_0(u) = \exp\left(-\frac{c}{a+d} u \right) \qquad u \geqslant 0 \qquad \text{and} \qquad g_k(u) = \frac{\mathrm{d}}{\mathrm{d}u}\left(2\frac{g_{k-1}(u)}{u+|x|+|y|} \right).$$

Finally the asymptotic expansion for $n = 2$ is

$$\hat{P}_t(x, y) \sim (4\pi \mathrm{i}t)^{-1} \exp\left[\frac{\mathrm{i}\,|x-y|^2}{4t} \right] + (4\pi \mathrm{i}t|x||y|)^{-1/2} \exp\left[\frac{\mathrm{i}(|x|+|y|)^2}{4t} \right]$$
$$\times \left\{ \gamma \tilde{K}_0\left(\frac{|x||y|}{2\mathrm{i}t} \right) + \sum_{k=1}^{n} \left(\frac{4\mathrm{i}t}{(|x|+|y|)^2} \right)^k \left[\frac{(-1)^k}{k!} \tilde{K}_0\left(\frac{|x||y|}{2\mathrm{i}t} \right) \tilde{w}_{k,k}(\gamma) \right.\right.$$
$$\left.\left. + \frac{(|x|+|y|)^2}{2|x||y|} \sum_{j=1}^{k-1} \frac{(-1)^{j-1}}{(j-1)!\,(k-j)!} \tilde{\psi}_{k-j}\left(\frac{|x||y|}{2\mathrm{i}t} \right) \tilde{w}_{k-1,j-1}(\gamma) \right] \right\}$$

where

$$\gamma = \left(\alpha - 2\ln \frac{t}{|x|+|y|} \right)^{-1} \qquad \tilde{K}_0(z) = \mathrm{e}^z \sqrt{\frac{2z}{\pi}} K_0(z)$$
$$\tilde{\psi}_m(z) = z^{m+1} \tilde{K}_0(z) \qquad m \geqslant 1$$

and

$$\tilde{w}_{k,j}(\gamma) =: \int_0^\infty \mathrm{e}^{-u/\gamma} w_{k,j}(u)\,\mathrm{d}u = \sum_{s=0}^{k+j} \gamma^{s+1} w_{k,j}^{(s)}(0) \tag{32}$$

is a polynomial (of order $k+j+1$) in $|\ln t^2/A|^{-1}$. Here, $w_{k,j}(u) = (u)_k \phi_j(u)$ is a polynomial of order $k+j$ ($w_{k,j}(0) = 0$ for $k \geqslant 1$) and $\phi_0(u) = 1$, $\phi_k(u) = \phi_{k-1}(u)(u + (2k-1)/2)$, $(u)_0 = 1$ and $(u)_{k+1} = (u+k)(u)_k$ for $k \in \mathbb{N}$.

We conclude with some remarks. The point interactions in dimension $n = 2$ present some specific features with respect to the cases $n = 1$ or $n = 3$. For instance, the time-dependent propagator $P_t(x, y)$ does not seem to possess a simple (single) integral representation. Also, the small-time asymptotic expansion of $P(t; x, y)$ is quite unusual. It contains terms like $(\alpha - 2\ln(t/(|x| + |y|)))^{-1}$, which are not present for $n = 1$ or $n = 3$ (neither for $n = 2$ for the free case).

This may relate to the fact that for $n = 2$ we *always* have a bound state (except the free case), while for $n = 1$ and $n = 3$ a bound state occurs for roughly half the parameters. Another feature of $n = 2$ is that $\mathbb{R}^2$ with one point $(x = 0)$ 'marked', or removed, has the homotopy group $\pi_1 = \mathbb{Z}$. The physical significance is that the trajectories of material points fall into classes labelled by their winding number around $x = 0$. As a consequence the path integral calculation should lead to a representation of the propagator as infinite sum (over integers) with coefficients to be determined.

Such a calculation has been performed for the Aharonov–Bohm effect in Morandi and Menossi (1984), see also Berry (1980) and Edwards (1967), where one has a charged particle in $\mathbb{R}^3$ plus a thin solenoid with magnetic flux Φ. As an idealization, the solenoid coincides with the third axis. Due to the symmetry, one can suppress the third coordinate and work in $\mathbb{R}^2 - \{0\}$, the point 0 representing the solenoid. The relevant coefficients in the sum provide a representation of the homotopy group and are equal to $\exp(2\pi i n\delta)$, $n \in Z$, where δ is proportional to Φ.

This particular magnetic interaction can be described by a free Hamiltonian, which is defined on multivalued wavefunctions obeying some non-periodic boundary conditions (depending on the flux) as the angle coordinate runs from 0 to 2π or, equivalently, on sections of a suitable line bundle over $\mathbb{R}^2 - \{0\}$. The point interactions discussed in this paper correspond to another physical problem described by a different Hamiltonian with a boundary condition for the radial coordinate at 0. The physical meaning is the 'strength' of the potential concentrated at 0. Such a potential causes of effects such as scattering, time delay etc. With respect to the Aharonov–Bohm situation, this could be interpreted, e.g., as the penetrability of the solenoid which produces the magnetic flux.

It will be interesting to combine these two distinct problems and study a general possible interaction between a particle and a 'point', representing, as an approximation, a point-like barrier (thin solenoid) together with a magnetic flux. Two of the 'coupling parameters' can be identified with the value of the flux and with the penetration coefficient of the solenoid, but in fact there should be a four-parameter family of self-adjoint extensions which, we believe, deserve further study. Similar remarks apply to the (idealized) double-slit experiment.

Acknowledgments

Some preliminary results reported in this work were presented at the most stimulating workshop organized by Gian Fausto Dell'Antonio at SISSA, Trieste, December 1992. We are very grateful to him and the participants, in particular Sergio Scarlatti and Sandro Teta, for interesting discussions.

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